

THE RANDOM GRAPH EMBEDS IN THE CURVE GRAPH OF ANY INFINITE GENUS SURFACE

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ABSTRACT. The Random Graph is an infinite graph with the universal property that any embedding of $G - v$ extends to an embedding of G , for any graph. In this paper we show that this graph embeds in the curve graph of any orientable infinite genus surface, showing that the curve system on an infinite genus surface is “as complicated as possible”.

1. INTRODUCTION

In this paper we will prove

Theorem 1.1. *The random graph embeds into the curve graph $\mathcal{C}(\Sigma)$, where Σ is the one-ended orientable surface of infinite genus without boundary.*

We adopt the terminology that an *embedding* of a graph $f : G \rightarrow H$ is a one-to-one map on the vertices so that (u, v) is an edge in G if and only if $(f(u), f(v))$ is an edge in H . (This is also called an *induced subgraph* elsewhere in the literature.)

The one-ended, infinite genus, orientable surface with one boundary component is a subsurface of any orientable infinite genus surface [Ric63]. The choice of a disk on Σ , the one-ended orientable surface of infinite genus without boundary, thus induces an embedding of the curve graph $\mathcal{C}(\Sigma)$ into the curve graph of an arbitrary orientable infinite genus surface. We conclude

Corollary 1.2. *The random graph embeds into the curve graph of any orientable infinite genus surface.*

Erdős and Rényi introduced the random graph from a probabilistic point of view, constructing a graph on countably many vertices by adding edges with probability $\frac{1}{2}$. The result of this construction is almost surely isomorphic to a unique object, which we call *the random graph* [ER63]. Rado gave an explicit construction of the random graph: Take as vertices the natural numbers \mathbb{N} . Given $x < y$, add an edge (x, y) if the x th bit of the binary expansion of y is 1 [Rad64]. The random graph enjoys a universal property, known as the extension property; for any graph G , if $G - v$ embeds into the random graph then this embedding can be extended to G .

The other graph of interest in this article is the curve graph of an infinite genus surface, with or without boundary. A simple closed curve on a surface is essential if no component of the complement is a disk, and non-peripheral if no component of the complement is an annulus. For brevity, we will use curve to mean the isotopy class of an essential non-peripheral simple closed curve. The intersection number of two curves (denoted $i(\alpha, \beta)$) is the minimum cardinality of the intersection taken over all realizations of α and β .

Fix a surface Σ . The curve graph $\mathcal{C}(\Sigma)$ has as vertices the curves on Σ , and an edge between the vertices corresponding to curves α and β when $i(\alpha, \beta) = 0$. (As an aside, this construction can be extended to a definition of a higher-dimensional simplicial complex of interest in its own right, but we will focus on the 1-skeleton [Har81, MM99, MM00].)

In fact, every finite graph G embeds in the curve graph for some closed surface of genus g . We outline a simple proof: Suppose G has n vertices. Let Σ_0 indicate a closed surface of genus large enough so that Σ_0 contains a collection of n curves in minimal position that pairwise intersect once¹, and identify these curves with the vertices of G arbitrarily. For each edge between a pair of vertices of G , add a handle near the intersection of the corresponding curves on Σ_0 , and thread one of the curves through the handle so that the new curves on the new surface do not intersect. The identification of the vertices of G with the resulting curve system extends to an embedding of G into the curve graph $\mathcal{C}(\Sigma)$ of the resulting surface Σ , by construction.

Remark 1.3. This leaves open the problem of determining the minimal genus such that every finite graph on n vertices embeds in the curve graph of that genus, a question communicated to us by Tarik Aougab.

This implies that every finite graph embeds into the curve graph of an infinite genus surface, and it suggests that this curve graph of an infinite genus surface should be quite complicated. Note, however, that this alone does not guarantee the presence of the random graph. Also note that it is apparent that the random graph does not embed in the curve graph of any finite genus surface: A complete subgraph of the curve graph of a closed surface of genus g has size at most $3g - 3$, whereas every finite graph embeds in the random graph.

Moreover, note that the curve graph of an infinite genus surface cannot itself be isomorphic to the random graph: Fix an infinite genus surface Σ . Pick a separating curve² γ and two curves α, β , one in each component of $\Sigma \setminus \gamma$. Let G be the graph in figure 1. We can embed $G - v$ into $\mathcal{C}(\Sigma)$ according to the labelling in the figure, but an extension to v would imply the existence of a curve disjoint from γ that intersects both α and β , a contradiction since γ is separating.

2. PROOF OF THEOREM 1.1

We will provide an explicit construction of a family of curves on the one-ended orientable infinite genus surface whose intersection relation is exactly described by the random graph. Our approach is in two parts; first we will give a countable collection of multicurves with this property, then describe how to add handles to convert these multicurves to curves without changing the intersection relation or the homeomorphism type of the surface.

Rado's construction fits more naturally into the setting of multicurves, so we first define a multicurve complex $m\mathcal{C}(\Sigma)$ analogous to the curve complex. Let the vertices of $m\mathcal{C}(S)$ be finite multisets of disjoint curves. For multicurves $U, V \in m\mathcal{C}(S)$, let

¹In fact, $\lceil \frac{n-1}{2} \rceil$ suffices. Such a system of curves has been referred to as a *complete 1-system* in the literature.

²Such a curve always exists: Pick a pair of curves that intersect exactly once, and take a regular neighborhood of the union. The result is a separating curve that cuts off a one-holed torus subsurface of Σ .

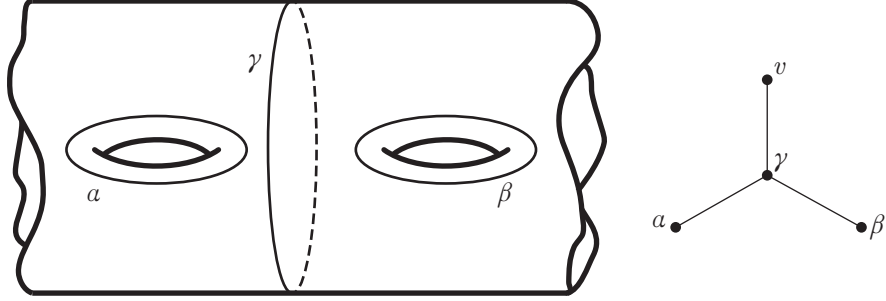


FIGURE 1. A collection of curves and a partial graph embedding that cannot extend.

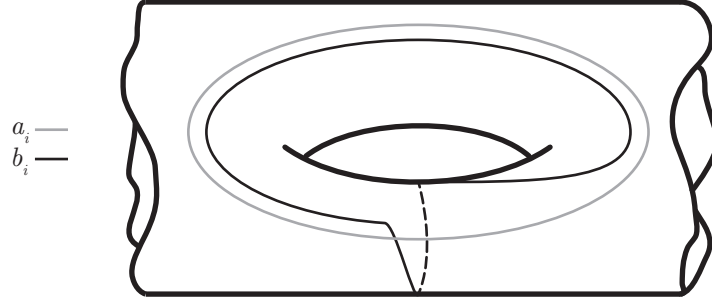
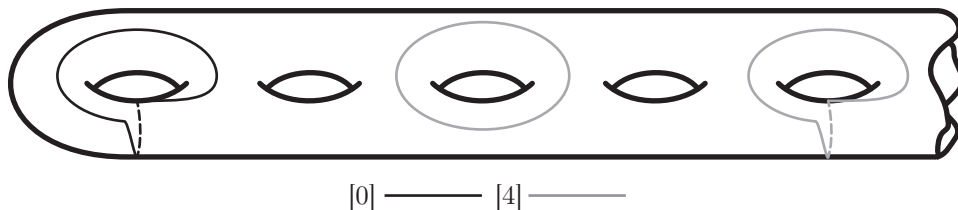


FIGURE 2. A possible choice for a_i and b_i at handle i .

$i(U, V)$ be the sum of intersection numbers $i(\alpha, \beta)$ over all $\alpha \in U, \beta \in V$. In analogy with the curve graph, there is an edge in $m\mathcal{C}(S)$ between U, V if $i(U, V) = 0$. (This multicurve graph seems unstudied in the literature, though the subgraph generated by a single homology class has been investigated [Irm12].) Below, we write multicurves additively, e.g. $3\alpha + \beta$ is the multicurve $\{\alpha, \alpha, \alpha, \beta\}$.

To fix notation, let Σ be the one-ended orientable surface of infinite genus. Note that the random graph is self-complementary (that is, the complement graph is isomorphic to the random graph), so we will work with the complement of Rado's model: Let $x, y \in \mathbb{N}$ with $x < y$ be adjacent when the x th bit in the binary expansion of y is 0. We describe below a map $[\cdot] : \mathbb{N} \rightarrow m\mathcal{C}(\Sigma)$ so that, for $x < y$, the intersection number $i([x], [y])$ is equal to the x th bit in the binary expansion of y . Such a map induces an embedding of the random graph into $m\mathcal{C}(\Sigma)$.

Realize Σ in \mathbb{R}^3 as the regular neighborhood of the lattice on points $\mathbb{N} \times \{0, 1\} \times \{0\}$. With this embedding the 'centers' of 'holes' of Σ occur at $(n + \frac{1}{2}, \frac{1}{2}, 0)$ with $n \in \mathbb{N}$. The intersection of Σ with the coordinate plane $\mathbb{R} \times \mathbb{R} \times \{0\}$ is the disjoint union of countably many circles and a real line. Let a_i be the circle component in the strip $(i - 1, i) \times \mathbb{R} \times \{0\}$, and let b_i be the Dehn twist of a_i around the intersection of the half-plane $\{i - \frac{1}{2}\} \times (-\infty, \frac{1}{2}) \times \mathbb{R}$ with Σ . In other words, a_i winds around the i th hole of Σ , and $i(a_i, b_j) = \delta_{i,j}$, as pictured in Figure 2.

FIGURE 3. An illustration of geometric realizations of $[0]$ and $[4]$.

Given a natural number x , let x_i be the i th binary digit in the expansion of x . We define

$$[x] = b_x + \sum_{i=0}^{\lceil \log_2 x \rceil} x_i \cdot a_i.$$

Figure 3 shows $[0]$ and $[4]$. By construction this is our desired map and the intersection relation among the multicurves $\{[n]\}_{n \in \mathbb{N}}$ is given by the random graph.

At this point one would like to blindly add handles to realize these multicurves as curves. However, for each bit there are infinitely many curves that need to be connected to the handle encoding that bit, so care must be exercised. Consider a new realization of Σ in \mathbb{R}^3 , as the regular neighborhood of the lattice on $\mathbb{N} \times \mathbb{N} \times \{0\}$. The centers of ‘holes’ are now at $(x + \frac{1}{2}, y + \frac{1}{2}, 0)$ for $x, y \in \mathbb{N}_{\geq 0}^2$. This naturally indexes the rows and columns of the embedding (row n is the regular neighborhood of points of the form $(x, n, 0)$, and the columns are similarly indexed). We take a_i, b_j as before (along the x -axis). For a multicurve $[x] = b_x + \sum_i x_i \cdot a_i$, construct the curve $c(x)$ by connecting each a_i or b_i in $[x]$ to row $x + 1$ by a pair of vertical lines along column i , and then connect these arcs to one another along the ‘back’ of Σ ; figure 4 shows $c(2)$ and $c(5)$. For $x < y$, we can realize $c(x)$ and $c(y)$ so that when x and y use a common column $c(x)$ passes outside of $c(y)$; hence $i(c(x), c(y)) = i([x], [y])$. (Note that, when curves intersect once, this intersection is necessarily essential [FM12].) We conclude that $\{c(n)\}_{n \in \mathbb{N}}$ is the vertex set of an embedding of the random graph in $\mathcal{C}(\Sigma)$.

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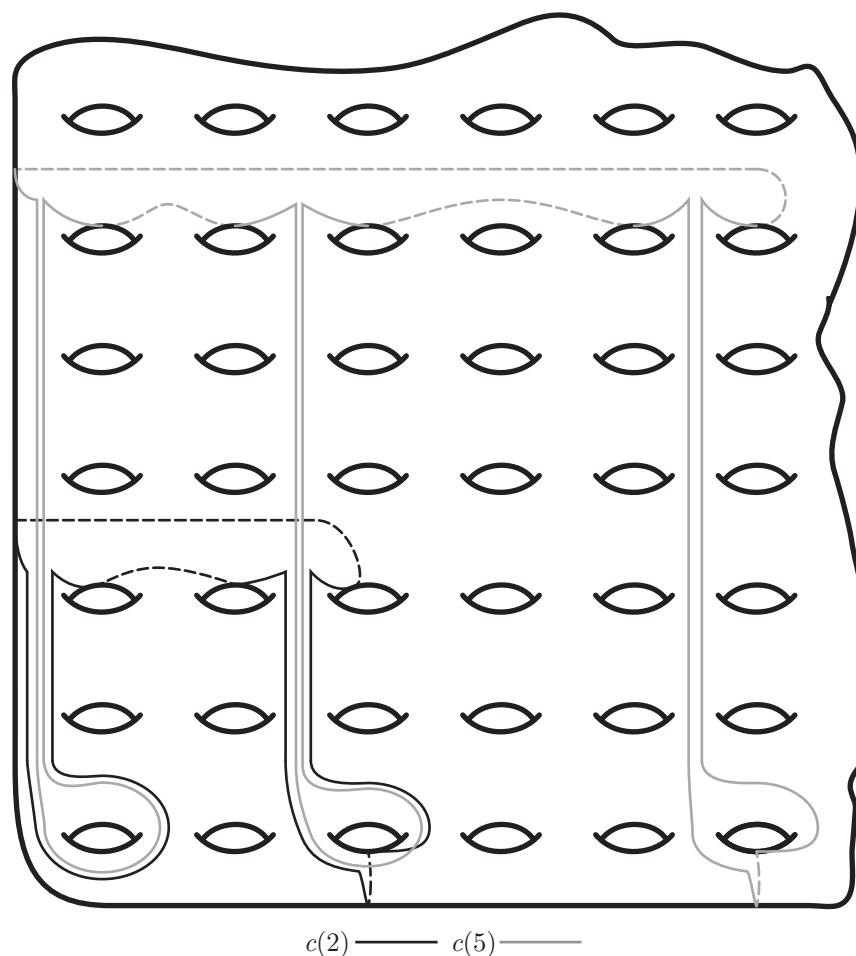


FIGURE 4. An illustration of $c(2)$ and $c(5)$ realizing $i(c(2), c(5)) = 1$.

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